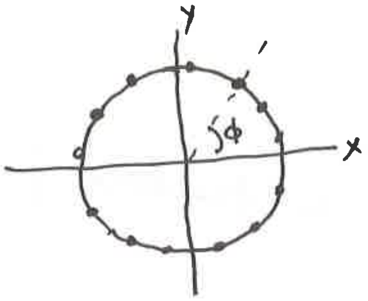
Lecture 3   IUCAA Refresher Course

Basics of GW theory   SK 19–25 May 2026

Effect of GW on a ring of particles.

Recall the relation $\delta L_\alpha = \frac{1}{2} h^{TT}_{\alpha\beta} L^\beta$ which was obtained by analysing the deviation eqn.



Consider a ring of particles in the xy plane.
A GW propagating in the z direction.

$$L^x_0 = \cos\phi \quad , \quad L^y_0 = \sin\phi .$$

We also have $\quad h^x_{\ x} = h_+(t) = - h^y_{\ y} = h_0 \cos\omega t$

$$\therefore \quad \delta L_x = \frac{1}{2} h_{xx} L^x_0 = \frac{1}{2} h_0 \cos\omega t \cos\phi .$$

$$\delta L_y = \frac{1}{2} h_{yy} L^y_0 = -\frac{1}{2} h_0 \cos\omega t \sin\phi$$

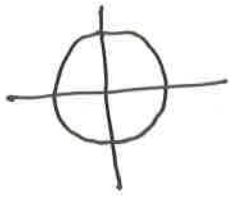
$$\therefore \quad L_x = L_x^0 + \delta L_x = \cos\phi + \frac{1}{2} h_0 \cos\omega t \cos\phi .$$

$$L_y = L_y^0 + \delta L_y = \sin\phi - \frac{1}{2} h_0 \cos\omega t \sin\phi .$$

$$\therefore \quad \frac{L_x^2}{\left(1 + \frac{1}{2} h_0 \cos\omega t\right)^2} + \frac{L_y^2}{\left(1 - \frac{1}{2} h_0 \cos\omega t\right)^2} = 1 .$$

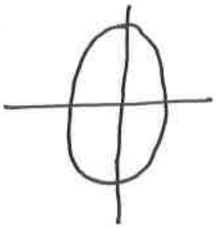
eqn. of an ellipse / circle.

$\omega t = \pi/2$



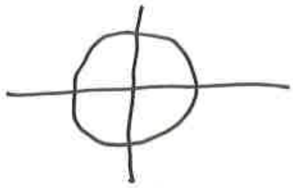
circle

$\omega t = \pi$

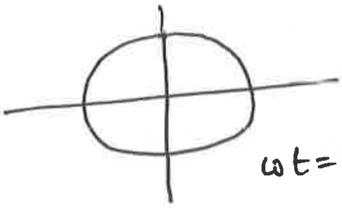


ellipse

$\omega t = 3\pi/2$



circle.



$\omega t = 2\pi$.

ellipse

and so on.

This picture repeats itself.

Comments

1 Everything depends on $h(t) \to$ its functional form

2 $\delta L^\alpha = \frac{1}{2} h^{\alpha\beta} L_\beta$ is the eqn that leads to the memory effect.

$h_+ (t \to \infty)$ different from $h_+ (t \to -\infty)$

Then δL^α gets a permanent change.

This is a residual change. Seen through the change in the deviation vector.

3 If, for some $h_{\alpha\beta}$ the circle permanently deforms to an ellipse we say that $h_{\alpha\beta}$ has memory, otherwise there is no memory in the GW signal.

Generation of GW – Quadrupole formula

$$\Box \bar{h}_{ij} = -\frac{16\pi G}{c^4} T_{ij}$$

How to solve the full inhomogeneous equ. for a given source. Use the Green's fn. method.

Translationally invariant Green's fn. for D'Alembertian operator $\longrightarrow$ $G(\underline{r}, \underline{r}') = -\dfrac{\delta(t' - t_r)}{|\underline{r} - \underline{r}'|\, 4\pi}$. $\quad t_r = t - \tilde{r}/c$. $\quad \tilde{r} = |\underline{r} - \underline{r}'|$.

$$\therefore \quad \bar{h}_{ij} = \frac{4G}{c^4} \int \frac{\delta(t' - t_r)\, dt'\, d^3r'}{|\underline{r} - \underline{r}'|} T_{ij}(t', \underline{r}')$$

$$= \frac{4G}{c^4} \int \frac{T_{ij}(t_r, \underline{r}')}{|\underline{r} - \underline{r}'|} d^3r'.$$

Far zone : $|\underline{r}|$ very large, ignore $\underline{r}'$ in denominator $\underline{r' \ll \lambda_{GW} \ll r}$

$$\bar{h}_{ij} = \frac{4G}{c^4 r} \int T_{ij}(t_r, \underline{r}')\, d^3r'.$$

We now need to evaluate this quantity.

Note that $\bar{h}_{00} = \dfrac{4G}{c^4 r} \displaystyle\int T_{00}\, d^3r' \Rightarrow \dfrac{M}{r}$ (Newtonian potential)

and $\bar{h}_{0\alpha} = \dfrac{4G}{c^4 r} \displaystyle\int T_{0\alpha}\, d^3r' \Rightarrow$ Momentum.

So we should focus on $\bar{h}_{\alpha\beta}$ only. After we evaluate we can use the projector to go over to the TT gauge :

$$P^{\alpha\beta}_{\gamma\delta} = \left(P^{\alpha}_{\gamma} P^{\beta}_{\delta} - \frac{1}{2} P_{\gamma\delta} P^{\alpha\beta}\right).$$

Let us now recall the conservation laws in Minkowski spacetime. This is questionable but it works. We use

$\partial_i T^{i\jmath} = 0.$

$\jmath = 0 \qquad \partial_0 T^{00} + \partial_\alpha T^{\alpha 0} = 0 \qquad (a)$

$\jmath = \beta \qquad \partial_0 T^{0\beta} + \partial_\alpha T^{\alpha\beta} = 0 \qquad (b)$

Take ∂_0 of (a) $\qquad \partial_0^2 T^{00} + \partial_0 \partial_\alpha T^{\alpha 0} = 0 \qquad (c)$

∂_β of (b) $\qquad \partial_\beta \partial_0 T^{0\beta} + \partial_\beta \partial_\alpha T^{\alpha\beta} = 0 \qquad (d)$

From (c) and (d) we get.

$$\partial_0^2 T^{00} = \partial_\beta \partial_\alpha T^{\alpha\beta}. \qquad (e)$$

We work with unprimed coordinates for the matters. We look at the LHS of (e). We will go back to primed later. We look at the RHS. Multiply the LHS by $x^\gamma x^\delta$ and so with the RHS.

We get $\qquad x^\gamma x^\delta \partial_0^2 T^{00} = x^\gamma x^\delta \partial_\beta \partial_\alpha T^{\alpha\beta}.$

$$\Rightarrow \quad \partial_0^2 \left(x^\gamma x^\delta T^{00}\right) = x^\gamma x^\delta \partial_\beta \partial_\alpha T^{\alpha\beta}.$$

Evaluate $\quad \partial_\beta \partial_\alpha \left[x^\gamma x^\delta T^{\alpha\beta}\right]$

$$= \partial_\beta \left[\delta^\gamma_\alpha x^\delta T^{\alpha\beta} + \delta^\delta_\alpha x^\gamma T^{\alpha\beta} + x^\gamma x^\delta \partial_\alpha T^{\alpha\beta}\right]$$

$$= \partial_\beta \left[x^\delta T^{\gamma\beta} + x^\gamma T^{\delta\beta} + x^\gamma x^\delta \partial_\alpha T^{\alpha\beta}\right]$$

$$= \delta^\delta_\beta T^{\gamma\beta} + x^\delta \partial_\beta T^{\gamma\beta} + \delta^\gamma_\beta T^{\delta\beta} + x^\gamma \partial_\beta T^{\delta\beta}$$

$$+ \delta^\gamma_\beta x^\delta \partial_\alpha T^{\alpha\beta} + x^\gamma \delta^\delta_\beta \partial_\alpha T^{\alpha\beta} + \underline{x^\gamma x^\delta \partial_\beta \partial_\alpha T^{\alpha\beta}}$$

$$= 2 T^{\gamma\delta} + 2 x^\delta \partial_\beta T^{\gamma\beta} + 2 x^\gamma \partial_\beta T^{\delta\beta} + x^\gamma x^\delta \partial_\beta \partial_\alpha T^{\alpha\beta}$$

$$\therefore \quad \partial_\beta \partial_\alpha \left[x^\gamma x^\delta T^{\alpha\beta} \right]$$

$$= 2T^{\gamma\delta} + 2x^\delta \partial_\beta T^{\gamma\beta} + 2x^\gamma \partial_\beta T^{\delta\beta} + \underline{x^\gamma x^\delta \partial_\beta \partial_\alpha T^{\alpha\beta}}$$

$$= 2T^{\gamma\delta} + 2\partial_\beta \left(x^\delta T^{\gamma\beta} \right) - 2T^{\gamma\delta}$$

$$+ 2\partial_\beta \left(x^\gamma T^{\delta\beta} \right) - 2T^{\gamma\delta} + x^\gamma x^\delta \partial_\beta \partial_\alpha T^{\alpha\beta}$$

$$= -2T^{\gamma\delta} + 2\partial_\beta \left(x^\delta T^{\gamma\beta} + x^\gamma T^{\delta\beta} \right)$$

$$+ x^\gamma x^\delta \partial_\beta \partial_\alpha T^{\alpha\beta}.$$

$$\therefore \quad x^\gamma x^\delta \partial_\beta \partial_\alpha T^{\alpha\beta} = \partial_\beta \partial_\alpha \left[x^\gamma x^\delta T^{\alpha\beta} \right]$$

$$+ 2T^{\gamma\delta} + 2\partial_\beta \left(x^\delta T^{\gamma\beta} + x^\gamma T^{\delta\beta} \right)$$

$$\therefore \quad \partial_0^2 \left(x^\gamma x^\delta T^{00} \right) = 2T^{\gamma\delta} + \partial_\beta \partial_\alpha \left[x^\gamma x^\delta T^{\alpha\beta} \right]$$

$$+ 2\partial_\beta \left(x^\delta T^{\gamma\beta} + x^\gamma T^{\delta\beta} \right)$$

Use this result in

$$\bar{h}_{\alpha\beta} = \frac{4G}{c^4 r} \int T_{\alpha\beta} \, d^3r'$$

$$\bar{h}^{\gamma\delta} = \frac{2G}{c^4 r} \int \partial_0^2 \left(x'^\gamma x'^\delta T_{00} \right) d^3r'$$

$$\bar{h}^{\gamma\delta} = \frac{2G}{c^4 r} \partial_0^2 \int \left(x'^\gamma x'^\delta T_{00} \right) d^3r'.$$

$$\bar{h}^{\gamma\delta} = \frac{2G}{c^4 r} \ddot{I}^{\gamma\delta}$$

where $I^{\gamma\delta} = \int x'^\gamma x'^\delta T_{00} \, d^3r'.$

quadrupole moment.

Taking the TT projection, one gets

$$\bar{h}_{\alpha\beta}^{TT} = \frac{2G}{c^4 r} \ddot{I}_{\alpha\beta}^{TT} \quad (1)$$

This is the famous quadrupole formula.

Check dimensions of RHS in (1): $\frac{M^{-1} L^3 T^{-2}}{L^4 T^{-4} \cdot L} \cdot M L^2 T^{-2} = M^0 L^0 T^0$

dimensionless as it should be.

The near zone solutions involves a multipole expansion of $\frac{1}{|\underline{r}-\underline{r}'|}$ $(r' \ll r \ll \lambda_{GW})$ In it, the $\frac{GM}{r}$ term is the Newtonian potential, Dipole disappears in the c.o.m. frame and the quadrupole involves the traceless $\mathcal{I}_{ij} = I_{ij}^{TT}$.

Therefore we get a connect between the near-zone and the far-zone result. We will reexpress using $\mathcal{I}_{ij}$.

Discuss other derivations of the quadrupole formula using slides.

(1) Dimensional arguments

(2) Pseudotensor

(3) Feynman graph derivation.

Geodesic eqn. analysis   Add to Lec 2 end pages

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0$$

$i=0$

$$\frac{d^2t}{d\tau^2} + \Gamma^t_{tt}\left(\frac{dt}{d\tau}\right)^2 + 2\Gamma^t_{t\alpha}\left(\frac{dt}{d\tau}\right)\left(\frac{dx^\alpha}{d\tau}\right) + \Gamma^t_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\frac{d^2t}{d\tau^2} + \Gamma^t_{tt}\left(\frac{dt}{d\tau}\right)^2 + 2\Gamma^t_{t\alpha}\left(\frac{dt}{d\tau}\right)^2 v^\alpha + \Gamma^t_{\alpha\beta}v^\alpha v^\beta\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\frac{d^2t}{d\tau^2} + \left(\frac{dt}{d\tau}\right)^2\left[\Gamma^t_{tt} + 2\Gamma^t_{t\alpha}v^\alpha + \Gamma^t_{\alpha\beta}v^\alpha v^\beta\right] = 0$$

$i=\alpha$

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{tt}\left(\frac{dt}{d\tau}\right)^2 + 2\Gamma^\alpha_{\beta t}\frac{dx^\beta}{d\tau}\cdot\frac{dt}{d\tau} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{tt}\left(\frac{dt}{d\tau}\right)^2 + 2\Gamma^\alpha_{\beta t}\frac{dx^\beta}{dt}\left(\frac{dt}{d\tau}\right)^2 + \Gamma^\alpha_{\beta\gamma}\left(\frac{dt}{d\tau}\right)^2\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} = 0$$

$$\frac{d^2x^\alpha}{d\tau^2} + \left(\frac{dt}{d\tau}\right)^2\left[\Gamma^\alpha_{tt} + 2\Gamma^\alpha_{\beta t}v^\beta + \Gamma^\alpha_{\beta\gamma}v^\beta v^\gamma\right] = 0$$

$$\frac{d^2x^\alpha}{d\tau^2} = \frac{d}{d\tau}\left[\frac{dx^\alpha}{dt}\frac{dt}{d\tau}\right] = \left\{\frac{d}{dt}\left[\frac{dx^\alpha}{dt}\frac{dt}{d\tau}\right]\right\}\frac{dt}{d\tau}$$

$$= \frac{d^2x^\alpha}{dt^2}\left(\frac{dt}{d\tau}\right)^2 + v^\alpha\frac{d^2t}{d\tau^2}\left(\frac{dt}{d\tau}\right)\frac{d\tau}{dt}$$

$$= \frac{d^2x^\alpha}{dt^2}\left(\frac{dt}{d\tau}\right)^2 + v^\alpha\frac{d^2t}{d\tau^2}.$$

$$\therefore \left(\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{tt} + 2\Gamma^\alpha_{\beta t}v^\beta + \Gamma^\alpha_{\beta\gamma}v^\beta v^\gamma\right)\left(\frac{dt}{d\tau}\right)^2$$

$$+ v^\alpha\left(-\left(\frac{dt}{d\tau}\right)^2\left\{\Gamma^t_{tt} + 2\Gamma^t_{t\alpha}v^\alpha + \Gamma^t_{\alpha\beta}v^\alpha v^\beta\right\}\right) = 0$$

Coordinate accl.

$$\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{tt} + 2\Gamma^\alpha_{\beta t}v^\beta + \Gamma^\alpha_{\beta\gamma}v^\beta v^\gamma - v^\alpha\Gamma^t_{tt} - 2\Gamma^t_{t\beta}v^\beta v^\alpha - \Gamma^t_{\beta\gamma}v^\beta v^\gamma v^\alpha = 0$$

In the TT gauge $\Gamma^{\alpha}_{tt} = \frac{1}{2}\eta^{\alpha\beta}\left[h_{\alpha t,t} + h_{\alpha t,t} - h_{tt,\alpha}\right]$

$$= 0$$

$\Rightarrow \quad \frac{d^2x^\alpha}{dt^2} = 0$ Coordinate acceleration is zero. (to linear order in h)

'Coordinates move with the wave'

Geodesic deviation note

Deviation L^α valid to linear order in 'L'.

Corrections will scale with $\frac{L}{\mathcal{L}}$

where $\mathcal{L}$ is the curvature scale.

For GW $\mathcal{L} \sim \lambda_{GW}$

$\therefore$ Corrections negligible for $\frac{L}{\lambda_{GW}} \ll 1$

or. $L \ll \lambda_{GW}$.

For LIGO $L \sim 4$ kms.

λ_{GW} for $\nu = 200$ Hz is

$$\lambda_{GW} = \frac{c}{\nu} = \frac{3\times10^8}{200} = 1.5\times10^6 \text{ m}$$

$$= 1500 \text{ km}.$$

So $\lambda_{GW} \gg L$.

Not so for space-based detectors like LISA

future.